ANISOTROPIC VECTOR FUNCTIONS OF VECTOR ARGUMENT

(ANIZOTROPNYE VEKTORNYE FUNKTSII VEKTORNOGO ARGUMENTA)

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The general form of anisotropic vector functions of vector argument, compatible with crystal symmetry, is derived. The desired functions $V^i = V^i(\mathrm{A})$ are represented as $V^i = \sum W^i_{(\mu)} f_\mu$, where $W^i_{(\mu)}$ are some fixed polynomials of the components of the vector A, whose specific form is indicated for each class of crystal symmetry, and f_μ are arbitrary functions of three functionally independent invariants of the vector A relative to the point group corresponding to this class. The obtained expansions satisfy the uniqueness and polynomial correspondence requirements. The former means that for given polynomials $W^i_{(\mu)}$ the functions f_μ are defined uniquely by the functions V^i; the latter means that the functions f_μ are polynomials of their arguments if the V^i are polynomials of the components of the vector A. These expansions are particularly convenient when the components of the vector-functions are polynomials of the components of the vector-argument.

1. Formulation of the problem. Let a vector field A^j, operate on a homogeneous anisotropic continuum with the result that another vector field

$$V^i = F^i(A^j) \qquad (1.1)$$

will originate in the medium.

The anisotropic functions F^i are not completely arbitrary; they should be compatible with the symmetry of the medium(*).

If the coordinate system is subjected to the transformation $x^i = x^i(x^{k'})$, where $\partial x^i / \partial x^{k'} = a^i_{\cdot k'}$, , the transformed components of the vectors A and V satisfy the relationship.

$$a^i_{\cdot k'} V^{k'} = F^i(a^i_{\cdot l'} A^{l'}) \qquad (1.2)$$

*) See [1] and [2] for definitions of isotropic and anisotropic tensor functions and their properties.

In particular, if the transformation $a^{i}_{\cdot k'}$ enters into the point symmetry group G of the medium, equalities (1.1) and (1.2) are equivalent. This is the necessary and sufficient condition for the compatibility of the vector function F^i with the symmetry of the medium.

Rivlin [3] indicated a method of construction of the functions F^i, which are compatible with the given point symmetry group G of the medium. All linear invariants $J_1, \ldots, J_l$, relative to $\mathbf{B}$ should be selected from the entire rational basis of invariants of two vectors $\mathbf{A}$ and $\mathbf{B}$ relative to the group G (it may be found in [4]). Any invariant of the vectors $\mathbf{A}$ and $\mathbf{B}$, which is linear in $\mathbf{B}$, may be represented as

$$\Psi = \sum_{\lambda=1}^{l} \Phi_\lambda J_\lambda \tag{1.3}$$

(where Φ_λ are invariants of the vector $\mathbf{A}$), and any vector function of the vector $\mathbf{A}$, which is compatible with the symmetry of the medium, as

$$F^i = \frac{\partial \Psi}{\partial B_i} = \sum_{\lambda=1}^{l} \Phi_\lambda \frac{\partial J_\lambda}{\partial B_i} \tag{1.4}$$

For practical application of anisotropic tensor functions it is important that the following two conditions be satisfied

1) Uniqueness - for a fixed choice of the invariants J_λ the functions Φ_λ should be defined uniquely by the functions F^i .

2) Polynomial correspondence - if F^i are polynomials of the components of the vector $\mathbf{A}$, the functions Φ_λ should also be polynomials of their arguments.

The functions $F^i(\mathbf{A})$, constructed by Rivlin [3], satisfy the second demand only. The aim herein is to construct anisotropic vector functions of a vector argument compatible with the crystal symmetry and satisfying both the mentioned demands(*).

2. Method of solution. As is known [6], an arbitrary invariant Φ of the vector $\mathbf{A}$ is written uniquely in one of three ways:

$$\begin{aligned} &(a) \qquad \Phi = f(\varphi_1, \varphi_2, \varphi_3) \\ &(b) \qquad \Phi = f_0(\varphi_1, \varphi_2, \varphi_3) + \psi f_1(\varphi_1, \varphi_2, \varphi_3) \\ &(c) \qquad \Phi = f_0(\varphi_1, \varphi_2, \varphi_3) + \sum_{\sigma=1}^{3} \psi_\sigma f_\sigma(\varphi_1, \varphi_2, \varphi_3) \end{aligned} \tag{2.1}$$

*) This problem has been solved earlier [5] for functions compatible with the texture symmetries.

Here $\varphi_1, \varphi_2, \varphi_3$ are functionally independent basis invariants of the vector $\mathbf{A}$, the so-called principal invariants; ψ, ψ_1, ψ_σ are complementary basis invariants, i.e., all the remaining invariants in the entire rational basis; f, f_0, f_1, f_σ are arbitrary functions of the principal invariants. Each of Formulas (2.1) is valid for the crystallographic classes of type (a), (b), (c), respectively(*).

Classes of type (a)

$1, m, mm2, mmm, 4mm, 4/mmm, 3m, 6mm, \bar{6}2m, 6/mmm, \bar{4}3m, m3m$

Classes of type (b)

$2, 2/m, 222, 4, 4/m, 422, \bar{4}2m, 3, 32, \bar{3}m, \bar{6}, 6, 6/m, 622, 23, m3, 432$

Classes of type (c)

$\bar{1}, \bar{4}, \bar{3}$

Formula (1.3) becomes

$$\text{(a)} \qquad \Psi = \sum_{\lambda=1}^{l} f_\lambda J_\lambda$$

$$\text{(b)} \qquad \Psi = \sum_{\lambda=1}^{l} (f_{\lambda,\,0} + \psi f_{\lambda,\,1}) J_\lambda \tag{2.2}$$

$$\text{(c)} \qquad \Psi = \sum_{\lambda=1}^{l} \left(f_{\lambda,0} + \sum_{\sigma=1}^{3} \psi_\sigma f_{\lambda,\,\sigma}\right) J_\lambda$$

These formulas may be combined

$$\Psi = \sum_{\lambda=1}^{L} \omega_\lambda f_\lambda \tag{2.3}$$

where the notation ω is used for any factor with f, whether it has the form J or ψJ. Evidently L equals $l, 2l, 4l$ in cases (a),(b) and (c), respectively.

Let f be an analytic function of its arguments. Then

$$f = \sum_{q=0}^{\infty} f_{(q)} = \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} R_{p_1p_2p_3} (\varphi_1)^{p_1} (\varphi_2)^{p_2} (\varphi_3)^{p_3} \tag{2.4}$$

Here $f_{(q)}$ is a polynomial in the principal invariants of the vector $\mathbf{A}$, whose degree in $\mathbf{A}$ is q; tne real coefficients $R_{p_1p_2p_3}$ are independent of $\mathbf{A}$. In this case even @ is represented as an infinite series $\Psi_{(0)} + \Psi_{(1)} + \Psi_{(2)}$ where $\Psi_{(q)}$ is a linear polynomial in $\mathbf{B}$ and of power q in @ Evidently

$$\Psi_{(q)} = P^i_{\cdot j_1 \ldots j_q} A^{j_1} \ldots A^{j_q} B_i, \qquad V_{(q)}{}^i = P^i_{\cdot j_1 \ldots j_q} A^{j_1} \ldots A^{j_q} \tag{2.5}$$

*) The international notation of classes of crystal symmetry (see [7], say) are used here and henceforth.

Here $P^{i}{}_{j_1 \ldots j_q}$ is a tensor which is once contravariant, q times covariant, symmetric in all the convariant induces and invariant relative to the group G. As is known [8], the number of independent components of such a tensor is

$$n_q = \frac{1}{N(G)} \sum_{g \in G} \chi_V(g) [\chi_V{}^q](g) \qquad (2.6)$$

where $N(G)$ is the order of the group G; $\chi_V(g)$ is the trace of the transformation matrix g from the group G; $[\chi_v{}^q](g)$ is the trace of the qth symmetric Kronecker power of the same matrix [9].

Let the degree of the principal invariants $\varphi_1, \varphi_2, \varphi_3$ relative to $\mathbf{A}$ be a_1, a_2, a_3. The homogeneous polynomial $f_{(q)}$ equals the sum of those members of the triple series (2.4) for which $p_1 a_1 + p_2 a_2 + p_3 a_3 = q$. The number of such members equals the denumerant [10]

$$D(q; a_1, a_2, a_3) = \frac{1}{q!} \left[\frac{d^q}{dt^q} \frac{1}{(1-t^{a_1})(1-t^{a_2})(1-t^{a_3})} \right]_{t=0} \qquad (2.7)$$

If there are r_0 invariants of zero degree in $\mathbf{A}$; r_1 invariants of first degree in $\mathbf{A}$, . . . , r_K invariants of degree K in $\mathbf{A}$, among the invariants ω_λ then the number of terms of degree q in $\mathbf{A}$ in (2.3) for Ψ equals

$$n_q{}^* = \sum_{k=0}^{\min(K,\, q)} r_k D(q-k; a_1, a_2, a_3) \qquad (2.8)$$

In general, $n_q{}^* > n_q$; this means that not all the members of the sum (2.3) are independent. Let us assume that there exists the expansion

$$\Psi = \omega_1 f_1 + \ldots + \omega_m f_m \qquad (2.9)$$

all of whose members are linearly independent while, however, satisfying the demand of polynomial correspondence as does (2.3).

Let s_k be the number of members of the form $\omega_\mu f_\mu$ in this expansion, in which the factor ω_λ is of degree k relative to $\mathbf{A}$. The numbers s_k should satisfy the infinite system of equations

$$\sum_{k=0}^{q} s_k D(q-k; a_1, a_2, a_3) = n_q \qquad (q = 0, 1, 2, \ldots) \qquad (2.10)$$

Taking into account that $D(0; a_1, a_2, a_3) = 1$, the solution of this system is easily written down in the recursion form

$$s_0 = n_0, \qquad s_q = n_q - \sum_{k=0}^{q-1} s_k D(q-k; a_1, a_2, a_3) \qquad (q = 1, 2, \ldots) \qquad (2.11)$$

On the other hand, $s_k \leqslant r_k$, and $s_k = 0$, in particular if $k > K$. It is hence sufficient to find $s_0, s_1, \ldots, s_K$.

When these numbers have been found, three cases are possible

$$\alpha) \quad r_q = s_q, \qquad \beta) \quad r_q > s_q = 0, \qquad \gamma) \quad r_q > s_q > 0$$

In case (α) all terms with factors ω of degree q in $\mathbf{A}$ remain in the expansion (2.10). In case (β) all such terms are discarded. In case (γ) it is necessary to discard $r_q - s_q$ such terms. Terms with such factors ω, as may be represented in the form $\omega = Q_1(\varphi_1, \varphi_2, \varphi_3)\omega_1 + \ldots + Q_m(\varphi_1, \varphi_2, \varphi_3)\omega_m$, where ω_μ are factors with remainder terms and $Q_\mu(\varphi_1, \varphi_2, \varphi_3)$ some polynomials of the principal invariants, should be discarded here.

The expansion (2.9) obtained after having discarded all excess members in the sum (2.3), and the corresponding Formula.

$$V^i = \sum_{\mu=1}^{m} \frac{\partial \omega_\mu}{\partial B_i} f_\mu \equiv \sum_{\mu=1}^{m} W_{(\mu)}{}^i f_\mu \tag{2.12}$$

solve the posed problem. We write (2.9) and (2.12) as

$$\Psi = \sum_{\mu=1}^{m} W_{(\mu)}{}^i B_i f_\mu, \qquad \mathbf{V} = \sum_{\mu=1}^{m} W_{(\mu)}{}^i \mathbf{e}_i f_\mu \tag{2.13}$$

The number of members in these sums is $m = s_0 + s_1 + \ldots + s_K$. On the other hand, a survey of the formulas presented in Section 3 shows that the number m equals 3,6,12, respectively, for groups of types (a), (b) and (c).

3. The functions V(A) compatible with crystal symmetry. Here the vector-functions of a vector argument in the form

$$\mathbf{V} = \sum_{\mu=1}^{m} W^i_{(\mu)} \mathbf{e}_i f_\mu$$

are listed here for all classes of crystal symmetry. The formulas are written in a rectangular Cartesian XYZ coordinate system with the directions $\mathbf{i}, \mathbf{j}, \mathbf{k}$. The orientation of the axes relative to the elements of crystal symmetry agrees with that proposed by the IRE (*), with the exception of classes 2, m and 2/ m, in which $2 \parallel Z$ or $m \perp Z$, and class $\bar{6}2m$, in which $2 \parallel X$, $\bar{6} \parallel Z$.

The classes for which the principal invariants of the vector are identical

*) See [7], Appendix 2 where the axes are denoted by Ox_1, Ox_2, Ox_3 respectively.

are combined in a series. The designation of that single class of type (a), which enters into the composition of the series, is ascribed to each series. The principal invariants of the vector A, i.e., the arguments of the arbitrary functions f_μ which agree for all classes of the given series, are indicated in parentheses after the designation of the series. For example, (3.2) means

$$\mathbf{V} = \mathbf{i}f_1(A_x, A_y, A_z^2) + \mathbf{j}f_2(A_x, A_y, A_z^2) + A_z\mathbf{k}f_3(A_x, A_y, A_z^2)$$

For brevity, the writing of the members which have already appeared in one of the preceding formulas is denoted by the symbols $\mathbf{V}(mm2)$, $\mathbf{V}(mmm)$ etc. For example, (3.4) should be read as

$$\mathbf{V} = \mathbf{k}f_1 + A_x\mathbf{i}f_2 + A_y\mathbf{j}f_3 + A_x\mathbf{j}f_4 + A_y\mathbf{i}f_5 + A_xA_y\mathbf{k}f_6$$

where $$f_\mu = f_\mu(A_z, A_x^2, A_y^2) \qquad (\mu = 1, \ldots, 6)$$

The symbol Σ in the formulas for the series $\bar{4}3m$ and $m3m$ means summation over the cyclic permutation of the subscripts x, y, z and the directions $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

Series 1 (A_x, A_y, A_z)

Class 1
$$\mathbf{V} = \mathbf{i}f_1 + \mathbf{j}f_2 + \mathbf{k}f_3 \tag{3.1}$$

Series m (A_x, A_y, A_z^2)

Class m
$$\mathbf{V} = \mathbf{i}f_1 + \mathbf{j}f_2 + A_z\mathbf{k}f_3 \tag{3.2}$$

Series $mm2$ (A_z, A_x^2, A_y^2)

Class $mm2$
$$\mathbf{V} = \mathbf{k}f_1 + A_x\mathbf{i}f_2 + A_y\mathbf{j}f_3 \tag{3.3}$$

Class 2
$$\mathbf{V} = \mathbf{V}(mm2) + A_x\mathbf{j}f_4 + A_y\mathbf{i}f_5 + A_xA_y\mathbf{k}f_6 \tag{3.4}$$

Series mmm (A_x^2, A_y^2, A_z^2)

Class mmm
$$\mathbf{V} = A_x\mathbf{i}f_1 + A_y\mathbf{j}f_2 + A_z\mathbf{k}f_3 \tag{3.5}$$

Class 222
$$\mathbf{V} = \mathbf{V}(mmm) + A_yA_z\mathbf{i}f_4 + A_zA_x\mathbf{j}f_5 + A_xA_y\mathbf{k}f_6 \tag{3.6}$$

Class $2/m$
$$\mathbf{V} = \mathbf{V}(mmm) + A_x\mathbf{j}f_4 + A_y\mathbf{i}f_5 + A_xA_yA_z\mathbf{k}f_6 \tag{3.7}$$

Class $\bar{1}$
$$\mathbf{V} = \mathbf{V}(2/m) + A_x\mathbf{k}f_7 + A_y\mathbf{k}f_8 + A_z\mathbf{i}f_9 + A_z\mathbf{j}f_{10} + A_xA_yA_z\mathbf{i}f_{11} + A_xA_yA_z\mathbf{j}f_{12} \tag{3.8}$$

Series $4mm\ (A_z,\ A_x^2+A_y^2,\ A_x^2A_y^2)$

Class $4mm$

$$\mathbf{V}=\mathbf{k}f_1+(A_x\mathbf{i}+A_y\mathbf{j})f_2+A_xA_y(A_x\mathbf{j}+A_y\mathbf{i})f_3 \qquad (3.9)$$

Class 4

$$\mathbf{V}=\mathbf{V}(4mm)+(A_x\mathbf{j}-A_y\mathbf{i})f_4+A_xA_y(A_x\mathbf{i}-A_y\mathbf{j})f_5+A_xA_y(A_x^2-A_y^2)\mathbf{k}f_6 \qquad (3.10)$$

Series $4/mmm\ (A_z^2,\ A_x^2+A_y^2,\ A_x^2A_y^2)$

Class $4/mmm$

$$\mathbf{V}=A_z\mathbf{k}f_1+(A_x\mathbf{i}+A_y\mathbf{j})f_2+A_xA_y(A_x\mathbf{j}+A_y\mathbf{i})f_3 \qquad (3.11)$$

Class 422 (3.12)

$$\mathbf{V}=\mathbf{V}(4/mmm)+A_z(A_x\mathbf{j}-A_y\mathbf{i})f_4+A_xA_y(A_x^2-A_y^2)\mathbf{k}f_5+A_xA_yA_z(A_x\mathbf{i}-A_y\mathbf{j})f_6$$

Class $\bar{4}2m$

$$\mathbf{V}=\mathbf{V}(4/mmm)+A_xA_y\mathbf{k}f_4+A_z(A_x\mathbf{j}+A_y\mathbf{i})f_5+A_xA_yA_z(A_x\mathbf{i}+A_y\mathbf{j})f_6 \qquad (3.13)$$

Class $4/m$

$$\mathbf{V}=\mathbf{V}(4/mmm)+(A_x\mathbf{j}-A_y\mathbf{i})f_4+A_xA_y(A_x\mathbf{i}-A_y\mathbf{j})f_5+A_xA_yA_z(A_x^2-A_y^2)\mathbf{k}f_6 \qquad (3.14)$$

Class $\bar{4}$

$$\mathbf{V}=\mathbf{V}(4/m)+(A_x^2-A_y^2)\mathbf{k}f_7+A_xA_y\mathbf{k}f_8+A_z(A_x\mathbf{i}-A_y\mathbf{j})f_9+$$
$$+A_z(A_x\mathbf{j}+A_y\mathbf{i})f_{10}+A_xA_yA_z(A_x\mathbf{i}+A_y\mathbf{j})f_{11}+A_xA_yA_z(A_x\mathbf{j}-A_y\mathbf{i})f_{12} \qquad (3.15)$$

Series $3m\ (A_z,\ A_x^2+A_y^2,\ A_y^3-3A_x^2A_y)$

Class $3m$

$$\mathbf{V}=\mathbf{k}f_1+(A_x\mathbf{i}+A_y\mathbf{j})f_2+(A_x^2\mathbf{j}-A_y^2\mathbf{j}+2A_xA_y\mathbf{i})f_3 \qquad (3.16)$$

Class 3 (3.17)

$$\mathbf{V}=\mathbf{V}(3m)+(A_x\mathbf{j}-A_y\mathbf{i})f_4+(A_x^2\mathbf{i}-A_y^2\mathbf{i}-2A_xA_y\mathbf{j})f_5+(A_x^3-3A_xA_y^2)\mathbf{k}f_6$$

Series $\bar{6}2m\ (A_z^2,\ A_x^2+A_y^2,\ A_x^3-3A_xA_y^2)$

Class $\bar{6}2\ m$

$$\mathbf{V}=A_z\mathbf{k}f_1+(A_x\mathbf{i}+A_y\mathbf{j})f_2+(A_x^2\mathbf{i}-A_y^2\mathbf{i}-2A_xA_y\mathbf{j})f_3 \qquad (3.18)$$

Class 32 (3.19)

$$\mathbf{V}=\mathbf{V}(\bar{6}2m)+A_z(A_x\mathbf{j}-A_y\mathbf{i})f_4+A_z(A_x^2\mathbf{j}-A_y^2\mathbf{j}+2A_xA_y\mathbf{i})f_5+(A_y^3-3A_x^2A_y)\mathbf{k}f_6$$

Class $\bar{6}$

$$\mathbf{V} = \mathbf{V}(\bar{6}2m) + (A_x\mathbf{j} - A_y\mathbf{i}) f_4 + (A_x^2\mathbf{j} - A_y^2\mathbf{j} + 2A_xA_y\mathbf{i}) f_5 + A_z(A_y^3 - 3A_x^2A_y)\mathbf{k} f_6 \tag{3.20}$$

Series $6mm$ $(A_z,\ A_x^2 + A_y^2,\ A_x^6 - 15A_x^4A_y^2 + 15A_x^2A_y^4 - A_y^6)$

Class $6mm$

$$\mathbf{V} = \mathbf{k} f_1 + (A_x\mathbf{i} + A_y\mathbf{j}) f_2 + (A_x^3 - 3A_xA_y^2)(A_x^2\mathbf{i} - A_y^2\mathbf{i} - 2A_xA_y\mathbf{j}) f_3 \tag{3.21}$$

Class 6

$$\mathbf{V} = \mathbf{V}(6mm) + (A_x\mathbf{j} - A_y\mathbf{i}) f_4 + (A_x^3 - 3A_xA_y^2)(A_x^2\mathbf{j} - A_y^2\mathbf{j} + 2A_xA_y\mathbf{i}) f_5 + (3A_x^5A_y - 10A_x^3A_y^3 + 3A_xA_y^5)\mathbf{k} f_6 \tag{3.22}$$

Series $6/mmm$ $(A_z^2,\ A_x + A_y^2,\ A_x^6 - 15A_x^4A_y^2 + 15A_x^2A_y^4 - A_y^6)$

Class $6/mmm$

$$\mathbf{V} = A_z\mathbf{k} f_1 + (A_x\mathbf{i} + A_y\mathbf{j}) f_2 + (A_x^3 - 3A_xA_y^2)(A_x^2\mathbf{i} - A_y^2\mathbf{i} - 2A_xA_y\mathbf{j}) f_3 \tag{3.23}$$

Class $6/m$

$$\mathbf{V} = \mathbf{V}(6/mmm) + (A_x\mathbf{j} - A_y\mathbf{i}) f_4 + (A_x^3 - 3A_xA_y^2)(A_x^2\mathbf{j} - A_y^2\mathbf{j} + 2A_xA_y\mathbf{i}) f_5 + A_z(3A_x^5A_y - 10A_x^3A_y^3 + 3A_xA_y^5)\mathbf{k} f_6 \tag{3.24}$$

Class 622

$$\mathbf{V} = \mathbf{V}(6/mmm) + A_z(A_x\mathbf{j} - A_y\mathbf{i}) f_4 + A_z(A_x^3 - 3A_xA_y^2)(A_x^2\mathbf{j} - A_y^2\mathbf{j} + 2A_xA_y\mathbf{i}) f_5 + (3A_x^5A_y - 10A_x^3A_y^3 + 3A_xA_y^5)\mathbf{k} f_6 \tag{3.25}$$

Class $\bar{3}m$

$$\mathbf{V} = \mathbf{V}(6/mmm) + A_z(A_x^2\mathbf{j} - A_y^2\mathbf{j} + 2A_xA_y\mathbf{i}) f_4 + (A_y^3 - 3A_x^2A_y)\mathbf{k} f_5 + A_z(A_y^3 - 3A_x^2A_y)(A_x\mathbf{i} + A_y\mathbf{j}) f_6 \tag{3.26}$$

Class $\bar{3}$

$$\mathbf{V} = \mathbf{V}(\bar{3}m) + (A_x\mathbf{j} - A_y\mathbf{i}) f_7 + (A_x^3 - 3A_xA_y^2)\mathbf{k} f_8 + A_z(A_x^2\mathbf{i} - A_y^2\mathbf{i} - 2A_xA_y\mathbf{j}) f_9 + (A_x^3 - 3A_xA_y^2)(A_x^2\mathbf{j} - A_y^2\mathbf{j} + 2A_xA_y\mathbf{i}) f_{10} + A_z(A_x^3 - 3A_xA_y^2)(A_x\mathbf{i} + A_y\mathbf{j}) f_{11} + A_z(3A_x^5A_y - 10A_x^3A_y^3 + 3A_xA_y^5)\mathbf{k} f_{12} \tag{3.27}$$

Series $\bar{4}3m$ $(\Sigma A_x^2,\ A_xA_yA_z,\ \Sigma A_y^2A_z^2)$

Class $\bar{4}3m$

$$\mathbf{V} = f_1\Sigma A_x\mathbf{i} + f_2\Sigma A_yA_z\mathbf{i} + f_3\Sigma A_x^3\mathbf{i} \tag{3.28}$$

Class 23

$$\mathbf{V} = \mathbf{V}(\bar{4}3m) + f_4\Sigma(A_y^2 - A_z^2)A_x\mathbf{i} + f_5\Sigma(A_y^2 - A_z^2)A_yA_z\mathbf{i} + f_6\Sigma(A_y^2 - A_z^2)A_x^3\mathbf{i} \tag{3.29}$$

Series $m3m$ $(\Sigma A_x^2, \Sigma A_y^2 A_z^2, A_x^2 A_y^2 A_z^2)$

Class $m3m$

$$\mathbf{V} = f_1 \Sigma A_x \mathbf{i} + f_2 \Sigma A_x^3 \mathbf{i} + A_x A_y A_z f_3 \Sigma A_y A_z \mathbf{i} \tag{3.30}$$

Class $m3$

$$\mathbf{V} = \mathbf{V}(m3m) + f_4 \Sigma (A_y^2 - A_z^2) A_x \mathbf{i} + \\ + f_5 \Sigma (A_y^2 - A_z^2) A_x^3 \mathbf{i} + [\Sigma A_x^4 (A_y^2 - A_z^2)] f_6 \Sigma A_x \mathbf{i} \tag{3.31}$$

Class 432

$$\mathbf{V} = \mathbf{V}(m3m) + f_4 \Sigma (A_y^2 - A_z^2) A_y A_z \mathbf{i} + \\ + A_x A_y A_z f_5 \Sigma (A_y^2 - A_z^2) A_x \mathbf{i} + A_x A_y A_z f_6 \Sigma (A_y^2 - A_z^2) A_x^3 \mathbf{i} \tag{3.32}$$

4. Verification of the solution. Having constructed expansions of the form (2.9) for each class of crystal symmetry (formulas (3.1)-(3.32), it is necessary to verify whether they satisfy the demands of uniqueness and polynomial correspondence, i.e., whether they are solutions of the problem.

The uniqueness of the obtained modes of writing the vector functions $V^i = V^i(\mathbf{A})$ is manifest in that the functions of the principal invariants f_μ in (2.13) are expressed uniquely in terms of the functions $V^i(\mathbf{A})$. . Let us examine the group(*) of each type separately.

If the crystallographic group G belongs to type (a), the proof of uniqueness reduces to evaluation of the Jacobian $\partial J_\lambda / \partial B_i$, since $\omega_\lambda = J_\lambda$ and $\lambda = 1, 2, 3$, in this case. It is easy to verify by a direct computation that all such Jacobians for the type (a) groups are not zero (see Section 3).

If the group G belongs to type (b) or (c), , let us introduce a type (a) group G^* into the considerations, whose principal invariants agree with the principal invariants of the group G.. Each type (a) group in the list of Section 3 is a group G^* for the remaining groups in the same series.

A group G, belonging to type (b), is an invariant subgroup of index 2 of its group G^* . Let g be an element of the group G^*, not in G, and $g^\wedge \mathbf{A}$ and $g^\wedge W^i_{(\mu)}$ the results of the transformation $g^\wedge$, corresponding to the element g, operating on the vector A and the polynomial $W^i_{(\mu)}$. Evidently, $g^\wedge f = f$. Hence, from (1.2) we obtain a system of six linear

*) The word "group" here means the point group corresponding to a given class of crystal symmetry.

equations in f_μ

$$\sum_{\mu=1}^{6} W^i_{(\mu)} f_\mu = V^i(\mathrm{A}), \qquad \sum_{\mu=1}^{6} g\,\hat{}\, W^i_{(\mu)} f_\mu = V^i(g\,\hat{}\,\mathrm{A}) \tag{4.1}$$

For this system to be solvable uniquely, its determinant should not be zero, and this indeed holds for all type (b) groups.

A group G, belonging to type (c), is an invariant subgroup of index 4 of its G^* group. Let us construct adjacent classes of the group G^* by means of the subgroup G and let us select elements g_1, g_2, g_3 from the different adjacent classes. Then f_μ is determined from the 12 linear equations

$$\sum_{\mu=1}^{12} W^i_{(\mu)} f_\mu = V^i(\mathrm{A}), \qquad \sum_{\mu=1}^{12} g_\sigma\,\hat{}\, W^i_{(\mu)} f_\mu = V^i(g_\sigma\,\hat{}\,\mathrm{A}) \qquad (\sigma = 1, 2, 3) \tag{4.2}$$

The proof of uniqueness reduces to verifying that the determinant of this system is non-zero.

Since Equations (2.10) are satisfied, the polynomial-correspondence requirement is fulfilled if all members of degree q in A are linearly independent for any q. It is evidently sufficient to prove that the members $\omega_\mu f_\mu$ in the expansion (2.9) are linearly independent for any choice of the f_μ (understandably, it is assumed that none of these functions is identically zero). But it is easy to see that the linear independence of the functions $\omega_\mu f_\mu$, follows for $f_\mu \neq 0$ from the fact that the determinants of the systems of linear equations considered above are not zero. If one of the functions f_μ is identically zero, it is necessary to prove the linear independence of the remaining members. It follows from the fact that at least one of the corresponding $(m-1)$th order minors of the mentioned determinant is not zero. Then, we can also consider the case when one more function f_μ is identically zero (then $(m-2)$th order minors are examined of the $(m-1)$th order determinant), etc.

5. **Example.** Let us consider all the calculations in an example with the class $\bar{4}2m$. The XYZ Cartesian coordinate system is selected so that $Z \parallel \bar{4}$, $X \parallel 2$. We find the invariants

$$\varphi_1 = A_z^2, \qquad \varphi_2 = A_x^2 + A_y^2, \qquad \varphi_3 = A_x^2 A_y^2; \qquad \psi = A_x A_y A_z \tag{5.1}$$

in [4 and 6]

From [4] we write down the invariants of the vectors A and B, linear in B,

$$J_1 = A_z B_z, \qquad J_2 = A_x B_x + A_y B_y, \qquad J_3 = A_x A_y B_z$$
$$J_4 = A_z(A_x B_y + A_y B_x), \qquad J_5 = A_x A_y (A_x B_y + A_y B_x) \tag{5.2}$$

The class $\bar{4}2m$ belongs to type (b); the degrees of the principal invariants are 2,2,4. From (5.1) and (5.2) we obtain $K = 6$. We find the denumerants $D(q; 2, 2, 4)$, henceforth denoted by D_q, by a direct computation; the numbers r_q are determined from (5.1) and (5.2); n_q^* are evaluated in (2.9); the n_q are written down from [11]; the s_q are evaluated from (2.12). All the computations should be carried out to $q = 6$. The obtained results are presented here in the table.

$q =$	0	1	2	3	4	5	6
$D_q =$	1	0	2	0	4	0	6
$r_q =$	0	2	2	1	2	2	1
$n_q^* =$	0	2	2	5	6	12	13
$n_q =$	0	2	2	5	5	10	10
$s_q =$	0	2	2	1	1	0	0

It hence follows that all the invariants ω of degree $0, 1, 2, 3$ in A (together with the functions f, by which they are multiplied) remain in the final expression for the invariant Ψ;; all the invariants ω of degrees 5 and 6 should be discarded; moreover, it is necessary to discard one of the fourth degree invariants: either ψJ_1 or ψJ_2. Since $\psi J_1 = \varphi_1 J_3$, precisely this invariant should be discarded. We finally obtain

$$\Psi = f_1 J_1 + f_2 J_2 + f_3 J_3 + f_4 J_4 + f_5 J_5 + f_6 \psi J_2 \tag{5.3}$$

This agrees with (3.13). According to (2.13), the desired vector-function $\mathbf{V}(\mathbf{A})$ is

$$\begin{aligned} V_x &= A_x f_2 + A_y A_z f_4 + A_x A_y^2 f_5 + A_x^2 A_y A_z f_6 \\ V_y &= A_y f_2 + A_x A_z f_4 + A_x^2 A_y f_5 + A_x A_y^2 A_z f_6 \\ V_z &= A_z f_1 + A_x A_y f_3 \end{aligned} \tag{5.4}$$

Here $4/mmm$ plays the part of the group G^*. Reflection in a plane perpendicular to the principal axis, say, $g = m_z$, may be selected as the element g, in the group $4/mmm$, but not in the group $\bar{4}2m$.. Evidently $\mathbf{V}(m_z\mathbf{A}) = \mathbf{V}(A_x, A_y, -A_z)$ Let $\mathbf{V}(m_z\mathbf{A}) = \mathbf{V}^*$. Then the solution of (4.1) is written thus:

$$\begin{aligned} f_1 &= \frac{V_z - V_z^*}{2A_z}, & f_2 &= \frac{A_x(V_x + V_x^*) - A_y(V_y + V_y^*)}{2(A_x^2 - A_y^2)} \\ f_3 &= \frac{V_z + V_z^*}{2A_x A_y}, & f_4 &= \frac{A_x(V_y - V_y^*) - A_y(V_x - V_x^*)}{2A_z(A_x^2 - A_y^2)} \\ f_5 &= \frac{A_x(V_y + V_y^*) - A_y(V_x + V_x^*)}{2A_x A_y(A_x^2 - A_y^2)}, & f_6 &= \frac{A_x(V_x - V_x^*) - A_y(V_y - V_y^*)}{2A_x A_y A_z(A_x^2 - A_y^2)} \end{aligned} \tag{5.5}$$

6. Concluding remarks. The obtained expansions are a natural extension of the inscriptions used in the phenomenological theories of an anisotropic continuum for the functional dependence between two vectors

$$V^i = P^i + P^i_{\cdot j}A^j + P^i_{\cdot jk}A^jA^k + P^i_{\cdot jkl}A^jA^kA^l + \dots \quad (6.1)$$

The tensors $P^i, P^i_{\cdot j}, P^i_{\cdot jk}, P^i_{\cdot jkl}, \dots$ are material tensors; they describe the properties of the medium and should be invariant relative to its point group G. .

In the generalization constructed here the properties of the medium are described by the functions f_μ. . The value of the uniqueness requirement is that only if it is satisfied does it become possible to compare the properties of different materials. Hence, any method of writing the anisotropic tensor functions of given symmetry may find practical application only under the condition that it satisfies the uniqueness requirement.

On the other hand, the polynomial-correspondence requirement has meaning only if it is assumed in advance that the vector V is a whole rational function of the vector A., Compliance with this requirement permits finding all the tensors P at once, in particular: (6.2)

$$P^i = V^i(0), \quad P^i_{\cdot j} = \frac{\partial V^i(0)}{\partial A^j}, \quad P^i_{\cdot jk} = \frac{1}{2!}\frac{\partial^2 V^i(0)}{\partial A^j \partial A^k}, \quad P^i_{\cdot jkl} = \frac{1}{3!}\frac{\partial^3 V^i(0)}{\partial A^j\,\partial A^k\,\partial A^l}, \dots$$

a problem whose solution by another method would be very tedious for a tensor P of high rank. In many theories of an anisotropic continuum it is actually assumed that the vector V is expanded in a series (6.1) in powers of the components of the vector A, and then the method developed herein of writing the vector functions V (A) is completely natural, and apparently most convenient.

However, in some cases anisotropic functions must be considered which it is not possible or convenient to expand in a series of the form (6.1) (discontinuous functions, functions with discontinuous derivatives, etc.). The anisotropic vector functions constructed here may be used even in these cases. Then f_μ in (2.9), (2.13) and (3.1) to (3.32) may be considered arbitrary single-valued functions of their arguments. As before, the uniqueness property is satisfied since the assumption that the f_μ are polynomials of their arguments is used only in the proof of the polynomial-correspondence, but not in the proof of the uniqueness. Thus, the method developed for writing the anisotropic vector functions is applicable even in these more general theories, but it is now impossible to consider it either natural or most convenient. For the mentioned theories it is more convenient at once to get rid of the polynomial-correspondence requirement by replacing it with the demand that the desired vector function be represented as the sum of three linearly independent (in the geometric but not the functional sense as in Sections 2 and 4) vector functions. The idea of such a mode of writing arbitrary anisotropic functions has been expressed in [1 and 2]; its specific development might be the subject of a separate publication.

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